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2 + 1 kinematical expansions: from Galilei to de Sitter algebras

Francisco J Herranz† and Mariano Santander‡

† Departamento de Física, Escuela Politécnica Superior, Universidad de Burgos, E-09006 Burgos, Spain

‡ Departamento de Física Teórica, Facultad de Ciencias, Universidad de Valladolid, E-47011 Valladolid, Spain

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Abstract. Expansion of a Lie algebra is the opposite process to contraction. Starting from a Lie algebra, the expansion process goes to another algebra, which is non-isomorphic and less Abelian. We propose an expansion method based on the Casimir invariants of the initial and expanded algebras and where the free parameters involved in the expansion are the curvatures of their associated homogeneous spaces. This method is applied for expansions within the family of Lie algebras of three-dimensional spaces and (2 + 1)D kinematical algebras. We show that these expansions are classed into two types. The first type makes the curvature of space or spacetime different from zero (i.e. it introduces a space or universe radius), while the other has a similar interpretation for the curvature of the space of worldlines, which is non-positive and equal to $-1/c^2$ in the kinematical algebras. We obtain expansions which go from Galilei to either Newton–Hooke or Poincaré algebras, and from these to de Sitter algebras, as well as some other examples.

1. Introduction

The concept of contraction of Lie algebras and groups arose in the study of the limit from relativistic to classical mechanics. As is well known, when the velocity of light goes to infinity the Poincaré group leads formally to the Galilei one. This idea, proposed and studied by Inönü and Wigner [1] also appeared in Segal [2] and was later developed by Saletan [3]. More recently, other approaches to the study of contractions, such as the graded contraction theory [4, 5] and the generalized Inönü–Wigner contractions [6] have been introduced. In general, a Lie algebra contraction starts from some Lie algebra and makes some non-zero structure constants vanish, giving rise to another Lie algebra which is more Abelian than the original one. The theory of graded contractions includes Inönü–Wigner contractions but goes beyond that and, for instance, may also relate different real forms of semisimple Lie algebras.

The opposite process of a contraction limit is generically, and rather imprecisely called an *expansion*. One specific way to implement the expansion idea is to replace the generators of the initial algebra by some functions of them; if the new generators close a Lie algebra then we have obtained an expansion of the original algebra [7]. This kind of process produces, so to speak, some non-zero structure constants which were previously equal to zero, in such a manner that the final algebra is *less* Abelian than the initial one. In this approach usually the expanded algebra is realized as a subalgebra within an irreducible representation of the universal enveloping algebra for the initial algebra. We remark that in the literature algebra expansions are also called algebra deformations and indeed this kind of process can be seen as a ‘classical deformation’. However, we will use the former name in order to avoid confusion with quantum algebra deformations.

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Unlike the study of Lie algebra contractions, the theory of expansions has not been systematized. Known expansions are those going from the inhomogeneous pseudo-orthogonal algebras $iso(p,q)$ to the semisimple ones $so(p+1,q)$ and similar expansions for the unitary algebras from $iu(p,q)$ to $u(p+1,q)$ [7, 8]. On the other hand, a different procedure which allows one to perform expansions $t_{qp}(so(p)\oplus so(q)) \to so(p,q)$ or $t_{qp}(u(p)\oplus u(q)) \to u(p,q)$ as well as their symplectic analogous has been introduced in [9] (see also references therein).

The set of quasi-orthogonal algebras [10] appears as a natural frame for developing a study of expansions, with a good balance between generality and suitability as an adapted tool for specific purposes. This set of algebras includes all pseudo-orthogonal algebras as well as a large number of graded contractions—relative to a given maximal fine grading—of the simple (pseudo)-orthogonal algebras. These contractions are, however, not the most general ones, but still somehow keep the properties linked to simplicity, an important fact which makes these algebras a natural subset among all graded contractions of the orthogonal algebras. When turning to expansions, these remarks should be reversed: it is true that in principle *any* Lie algebra can be realized in the universal enveloping algebra of a direct product of Heisenberg algebras, as Schwinger realizations for the simple cases clearly show [11]. However, as in the most general set of contractions, it seems pertinent to restrict oneself to the study of expansions amongst quasi-simple algebras, which should reverse the contractions found in this family. For instance we would find, amongst many other expansions, the ones concerning the kinematical algebras [12]: expansions going from Newton–Hooke to de Sitter algebras, from Galilei to Newton–Hooke or to Poincaré algebras, further to expansions from Poincaré to the de Sitter or from Galilei to the two Newton–Hooke algebras. Note that the known transitions $iso(p,q) \to so(p+1,q)$ mentioned above include only expansions from Poincaré to de Sitter algebras, but not for the remaining quoted cases. As far as we know, specific possibilities for such general expansion schemes have not been studied in some generality; see, however, [13] for the $(1+1)$-dimensional case.

The aim of this paper is to provide a simple new expansion procedure and to apply it to the manageable but non-trivial case of the Lie algebras of motion groups in three-dimensional (3D) spaces. These expansions would include all the expansions of kinematical algebras in $(2+1)$ dimensions and a few more, non-kinematical examples. Thus in the next section we present the structure of the main kinematical algebras which include three 'absolute time' cases (two Newton–Hooke and Galilei) and three 'relative time' ones (two de Sitter and Poincaré). In section 3 we propose an expansion method which is based in the Casimir invariants of the two Lie algebras involved in the expansion. For instance, the initial Lie algebra may be related to a spacetime of curvature zero, while the expanded algebra corresponds to an homogeneous space with constant non-zero curvature; from this point of view we will see how the expansion process introduces the curvature as a *free* parameter. The remaining sections of the paper are devoted to analysing in detail all the possible kinematical expansions cast into two types: 'spacetime' expansions which starting from the algebra of a flat spacetime will introduce curvature in spacetime, and 'speedspace' expansions which recover a 'relative time' spacetime with a finite relativistic constant c (equal to the velocity of the light), starting from the algebra of an 'absolute time' (and hence $c=\infty$) spacetime.

2. The (2 + 1)-dimensional kinematical algebras

Let us consider an homogeneous spacetime with curvature κ and either of 'absolute time' type (formally described by letting $c\to\infty$) or 'relative time' one, with relativistic constant c. Let H, P_i, K_i $(i=1,2)$ and J the generators of time translations, space translations, boosts and spatial rotations, respectively. The structure of the kinematical algebras we are going to deal with can be written collectively in terms of two real coefficients $\omega_1 = \kappa$ and $\omega_2 = -1/c^2$ as follows:

$$\begin{array}{lll} [J, P_i] = \epsilon_{ij} P_j & [J, K_i] = \epsilon_{ij} K_j & [J, H] = 0 \\ [P_1, P_2] = \omega_1\omega_2 J & [K_1, K_2] = \omega_2 J & [P_i, K_j] = \delta_{ij}\omega_2 H \\ [H, P_i] = \omega_1 K_i & [H, K_i] = -P_i & i, j = 1, 2 \end{array} \qquad (2.1)$$

where ϵ_{ij} is a skew-symmetric tensor such that $\epsilon_{12} = 1$, $\epsilon_{21} = -1$ and $\epsilon_{11} = \epsilon_{22} = 0$. The main reason to introduce ω_2 instead of c is to allow *positive* values, whenever (2.1) does not have a kinematical interpretation, but nevertheless makes perfect sense as a Lie algebra. Thus each coefficient ω_i can take positive, negative or zero values, and the commutators (2.1) give rise to nine Lie algebras, which should be considered as different in this context. For each such algebra symmetric homogeneous space can be built up by taking the quotient by the subalgebra generated by K_i, J. These Lie algebras as well as the homogeneous spaces are displayed in table 1 according to the values of the pair (ω_1, ω_2).

Table 1. 3D isometry Lie algebras and their homogeneous spaces, including $(2+1)$D kinematical algebras.

$so(4)$ $(+, +)$ 3D Elliptic space	$\longrightarrow$	$iso(3)$ $(0, +)$ 3D Euclidean space	$\longleftarrow$	$so(3, 1)$ $(-, +)$ 3D Hyperbolic space
$\downarrow$		$\downarrow$		$\downarrow$
$t_4(so(2) \oplus so(2))$ $(+, 0)$ Oscillating NH $(2+1)$D spacetime	$\longrightarrow$	$iiso(2)$ $(0, 0)$ Galilean $(2+1)$D spacetime	$\longleftarrow$	$t_4(so(2) \oplus so(1, 1))$ $(-, 0)$ Expanding NH $(2+1)$D spacetime
$\uparrow$		$\uparrow$		$\uparrow$
$so(2, 2)$ $(+, -)$ Anti-de Sitter $(2+1)$D spacetime	$\longrightarrow$	$iso(2, 1)$ $(0, -)$ Minkowskian $(2+1)$D spacetime	$\longleftarrow$	$so(3, 1)$ $(-, -)$ de Sitter $(2+1)$D spacetime

The value of ω_2 can be thought of as related to the signature of the metric in the homogeneous space, which is definite positive for $\omega_2 > 0$ and indefinite (hence Lorentzian type in this 3D case) for $\omega_2 < 0$, with the Galilean degenerate metric which corresponds to 'absolute time' in the intermediate case $\omega_2 = 0$. Therefore, the three algebras of the first row with $\omega_2 > 0$ do not allow a *literal* interpretation in terms of a spacetime, and instead they are the Lie algebras of the motion groups of three-dimensional *Riemannian* spaces of constant curvature $\omega_1 = \kappa$. Kinematical algebras [12] arise whenever $\omega_2 \leqslant 0$, that is, when the boosts generate non-compact subgroups. The coefficient ω_1 is the universe curvature κ; the so-called universe radius R is related with ω_1 by either $\omega_1 = 1/R^2$ or $\omega_1 = -1/R^2$. The relativistic constant c plays a role analogous to R when ω_2 is *negative*, $\omega_2 = -1/c^2$. The three algebras of the second row (NH means Newton–Hooke) correspond to 'absolute time' spacetimes with $\omega_2 = 0$ or $c = \infty$, while those of the third row are associated to 'relative time' spacetimes with $\omega_2 < 0$ and a finite value for c.

These Lie algebras have two Casimir invariants given by

$$\begin{aligned} \mathcal{C}_1 &= \omega_2 H^2 + P_1^2 + P_2^2 + \omega_1(K_1^2 + K_2^2) + \omega_1\omega_2 J^2 \\ \mathcal{C}_2 &= \omega_2 H J - P_1 K_2 + P_2 K_1 \end{aligned} \qquad (2.2)$$

which in the kinematical cases $\omega_2 \leqslant 0$ correspond to the energy and angular momentum of a particle in the free kinematics of the spacetime corresponding to (ω_1, ω_2), respectively. When $\omega_2 > 0$ these expressions for the Casimirs cannot of course be interpreted in physical terms as energy and angular momentum.

We recall that each Lie algebra g of table 1 admits three involutive automorphisms, which we will name according to their natural interpretation in the kinematical case: parity $\mathcal{P}$, time-reversal $\mathcal{T}$ and their product $\mathcal{PT}$ defined by [12]

$$\begin{aligned} &\mathcal{P}: \qquad (H, P_i, K_i, J) \to (H, -P_i, -K_i, J) \\ &\mathcal{T}: \qquad (H, P_i, K_i, J) \to (-H, P_i, -K_i, J) \\ &\mathcal{PT}: \qquad (H, P_i, K_i, J) \to (-H, -P_i, K_i, J). \end{aligned} \tag{2.3}$$

These mappings clearly leave the Lie brackets (2.1) invariant. For further purposes we consider direct sum decompositions of g into anti-invariant and invariant generators under the action of the involutions $\mathcal{PT}$ and $\mathcal{P}$:

$$\begin{aligned} &\mathcal{PT}: \qquad g = p^{(1)} \oplus h^{(1)} \qquad p^{(1)} = \langle H, P_i \rangle \qquad h^{(1)} = \langle K_i, J \rangle \\ &\mathcal{P}: \qquad g = p^{(2)} \oplus h^{(2)} \qquad p^{(2)} = \langle P_i, K_i \rangle \qquad h^{(2)} = \langle H \rangle \oplus \langle J \rangle. \end{aligned} \tag{2.4}$$

Both are Cartan decompositions, verifying

$$[h^{(i)}, h^{(i)}] \subset h^{(i)} \qquad [h^{(i)}, p^{(i)}] \subset p^{(i)} \qquad [p^{(i)}, p^{(i)}] \subset h^{(i)}. \tag{2.5}$$

Note that $h^{(i)}$ is always a Lie subalgebra of g, while $p^{(i)}$ is only a subalgebra whenever $\omega_i = 0$ ($i = 1, 2$), and in that case it is Abelian: $[p^{(i)}, p^{(i)}] = 0$. Hence g is the Lie algebra of the motion group G of the following symmetrical homogeneous spaces:

$$\begin{aligned} &\mathcal{S}^{(1)} = G/H^{(1)} \qquad \dim(\mathcal{S}^{(1)}) = 3 \qquad \mathrm{curv}(\mathcal{S}^{(1)}) = \omega_1 \\ &\mathcal{S}^{(2)} = G/H^{(2)} \qquad \dim(\mathcal{S}^{(2)}) = 4 \qquad \mathrm{curv}(\mathcal{S}^{(2)}) = \omega_2 \end{aligned} \tag{2.6}$$

where $H^{(1)}$, $H^{(2)}$, the subgroups whose corresponding Lie algebras are $h^{(1)}$, $h^{(2)}$, are the isotropy subgroups of a point/event and a (timelike) line, respectively. Therefore, $\mathcal{S}^{(1)}$ is identified either to a three-dimensional space of points or to a (2 + 1)-dimensional spacetime, in both cases with constant curvature ω_1. Likewise, $\mathcal{S}^{(2)}$ is a four-dimensional space, whose 'points' can be identified with (timelike) lines in the former space; this has a natural connection and metric structure, whose curvature turns out to be 'constant' (in some suitable rank-two sense which is compatible with the fact that this space always contains a flat submanifold whose dimension equals to the rank) and equals ω_2.

To take the constant ω_1 (respectively ω_2) equal to zero is equivalent to perform an Inönü–Wigner contraction [1] starting from some algebra where $\omega_1 \neq 0$ (respectively, $\omega_2 \neq 0$). In this contraction, the invariant subalgebra is $h^{(1)}$ (respectively, $h^{(2)}$), while the remaining generators are multiplied by a parameter ε; the contracted algebra appears as the limit $\varepsilon \to 0$. If we perform this limiting procedure starting from the generic algebra (2.1) we find that a spacetime contraction makes to vanish the curvature ω_1 of $\mathcal{S}^{(1)}$ ($R \to \infty$), while a speedspace contraction makes zero the curvature ω_2 of $\mathcal{S}^{(2)}$ (in the kinematical case $c \to \infty$):

$$\begin{aligned} &\omega_1 \to 0: \quad \text{spacetime contraction} \quad (H, P_i, K_i, J) \to (\varepsilon H, \varepsilon P_i, K_i, J) \quad \varepsilon \to 0 \\ &\omega_2 \to 0: \quad \text{speedspace contraction} \quad (H, P_i, K_i, J) \to (H, \varepsilon P_i, \varepsilon K_i, J) \quad \varepsilon \to 0. \end{aligned} \tag{2.7}$$

In table 1 horizontal arrows correspond to spacetime contractions and the vertical ones to speedspace contractions. The Lie algebra expansions we are going to describe in the next sections are somewhat the opposite process and allow us to recover these constants starting from a contracted algebra. In geometrical terms, an expansion allows us to introduce curvature out of a flat space.

3. An expansion method

Let g and g' be two Lie algebras with commutation rules given by (2.1). We suppose that g is a contracted algebra obtained from g' by making zero *one* of the two constants ω_1 or ω_2, say ω_a, so that $g' \to g$ when $\omega_a \to 0$, while the other constant, (say ω_b) does not change. Now we want to consider the *opposite* situation: we look to g as the initial algebra, and we aim to recover g', which we shall call the *expanded* algebra, starting from g; for which we have to introduce a non-zero value for ω_a in some way. In the following we explain the expansion method we propose. The index a will always refer to the constant which is being 'expanded' from a zero value to a non-zero one in the expansion process.

Let $\mathcal{C}_1, \mathcal{C}_2$ be the two Casimirs of the initial Lie algebra g (with $\omega_a = 0$) and $\mathcal{C}'_1, \mathcal{C}'_2$ those of the final algebra g' (with $\omega_a \neq 0$ and the same remaining constant ω_b as g). A glance at the explicit expressions (2.2) clearly shows two facts: (a) $\mathcal{C}_i = \mathcal{C}'_i|_{\omega_a=0}$, and (b) $\mathcal{C}'_i$ is *linear* in the chosen ω_a. This suggests splitting each 'expanded' Casimir into two terms according to the presence of the constant ω_a. Obviously, the term independent of ω_a is just the 'contracted' Casimir, so these decompositions define, out of the formal expressions for the initial and the expanded Casimirs, some elements in the universal enveloping algebra of the initial algebra as

$$\mathcal{C}'_1 = \mathcal{C}_1 + \omega_a \mathcal{J}_1 \qquad \mathcal{C}'_2 = \mathcal{C}_2 + \omega_a \mathcal{J}_2 \tag{3.1}$$

where ω_a does not appear in any of the terms $\mathcal{C}_l$, $\mathcal{J}_l$ ($l = 1, 2$). We now consider the linear combination

$$\mathcal{J} = \alpha_1 \mathcal{J}_1 + \alpha_2 \mathcal{J}_2 \tag{3.2}$$

where α_1, α_2 are two constants to be determined and we will assume we are working in the universal enveloping algebra of g within an irreducible representation of g.

We define some elements in this universal enveloping algebra as the following functions of the generators X_k of g:

$$X'_k := \begin{cases} X_k & \text{if} \quad [\mathcal{J}, X_k] = 0 \\ [\mathcal{J}, X_k] & \text{if} \quad [\mathcal{J}, X_k] \neq 0. \end{cases} \tag{3.3}$$

The aim is to make these elements X'_k close a Lie algebra isomorphic to g'. Once $\mathcal{J}$ is given, the commutators of X'_k are completely determined, so that the only freedom at our disposal in this procedure lies in the choice of the constants α_l.

The computations of commutators of the new elements X'_k can be shortcut in some cases by use of the following result.

Proposition 1. *Suppose that the initial Lie algebra g with generators X_k has a direct sum decomposition as a vector space as $g = t \oplus k$ where k is the subalgebra determined by the condition $k = \langle X_k \mid [\mathcal{J}, X_k] = 0 \rangle$ and t is some vector subspace supplementary to k (notice that all elements in t do not commute with $\mathcal{J}$). Suppose also that for commutators of elements in k and t we have*

$$[k, k] \subset k \qquad [k, t] \subset t. \tag{3.4}$$

Then the generators X'_k defined by (3.3) for the expanded algebra g' have the 'same' Lie brackets $[k', k']$ and $[k', t']$ as the initial algebra g.

The proof is trivial for $[k', k']$ as the generators involved are invariant in the expansion and they directly span the Lie subalgebra k'. For $[k', t']$ we compute a generic Lie bracket between $X'_l \in k'$ and $X'_m \in t'$:

$$[X'_l, X'_m] = [X_l, \mathcal{J} X_m - X_m \mathcal{J}] = \mathcal{J}[X_l, X_m] - [X_l, X_m]\mathcal{J}. \tag{3.5}$$

As $[k, t] \subset t$, the commutator $[X_l, X_m] = C_{lm}^n X_n \in t$, so that

$$[X'_l, X'_m] = C_{lm}^n(\mathcal{J}X_n - X_n\mathcal{J}) = C_{lm}^n X'_n \in t'. \tag{3.6}$$

We remark that the decomposition in proposition 1 is defined in a way independent to the Cartan decompositions in (2.5), but it might coincide with them. In any case, the aim of the expansion idea is to find the commutation relations of the Lie algebra g' for the new generators X'_k. Whenever the hypotheses of proposition 1 are fulfilled, part of these commutation relations are automatically satisfied and to find the correct expanded commutation relations we only have to compute the brackets $[t', t']$ and to enforce for them the corresponding commutation relations of g'. In this way we obtain some equations involving α_l, $\mathcal{C}_l$ and ω_a; their solutions characterize the constants α_l. The coefficient ω_a (not appearing in g) is introduced in this last step.

In the next sections we apply this method to the algebras of table 1, reversing the direction of the contraction arrows. As we have two 'curvatures' we will consider two types of expansions: spacetime expansions, which out of $\omega_1 = 0$ recover ω_1, and speedspace expansions which similarly recover ω_2. In most cases the assumptions of proposition 1 will be satisfied, and for each expansion starting from $\omega_a = 0$ we will find that initially $[t, t] = 0$, and after the expansion $[t', t'] \subset k'$ due to the presence of an 'expanded' non-zero value for ω_a. It is remarkable that in the expansion which goes from Galilei to Poincaré it will be necessary to consider the initial Galilei algebra with a central extension; however, the procedure just described is still valid. Actually, this fact already happens in $(1+1)$ dimensions [13].

4. Spacetime expansions or ω_1 expansions

The purpose of this section is to discuss the expansions which starting from the algebra with $\omega_1 = 0$ 'introduce' a non-zero value for the constant ω_1. The value of ω_2 will remain unchanged in the expansion. Some details are slightly different according to either $\omega_2 \neq 0$ or $\omega_2 = 0$, so we will present these two cases separately. When applied to the kinematical algebras, this expansion leads from the Galilei algebra to the two Newton–Hooke ones, and from the Poincaré case to the two de Sitter algebras. In the non-kinematical case $\omega_2 > 0$ the expansion carries from 3D Euclidean algebra to either the elliptic or hyperbolic ones.

4.1. From Poincaré to de Sitter

We consider as initial algebras those with $\omega_1 = 0$ and $\omega_2 \neq 0$ which are the Euclidean $iso(3)$ (for $\omega_2 > 0$) and the Poincaré $iso(2, 1)$ (for $\omega_2 < 0$) algebras; they are associated with a flat 3D Euclidean space and to a relativistic flat $(2+1)$D spacetime, respectively. The Lie brackets which due to the initial condition $\omega_a \equiv \omega_1 = 0$ vanish in the general commutation relations (2.1) are

$$[P_1, P_2] = 0 \qquad [H, P_i] = 0 \tag{4.1}$$

and the two Casimirs (2.2) reduce to

$$\mathcal{C}_1 = \omega_2 H^2 + P_1^2 + P_2^2 \qquad \mathcal{C}_2 = \omega_2 HJ - P_1K_2 + P_2K_1. \tag{4.2}$$

The expansion to the $so(3)$ or $so(2, 1)$ algebras which correspond to $\omega_1 \neq 0$ and the same initial value for ω_2 requires replacing the three Lie brackets in (4.1) by those corresponding to $\omega_1 \neq 0$, which read

$$[P'_1, P'_2] = \omega_1\omega_2 J' \qquad [H', P'_i] = \omega_1 K'_i. \tag{4.3}$$

We split the Casimirs of the final semisimple algebras as

$$\begin{aligned} \mathcal{C}'_1 &= \mathcal{C}_1 + \omega_1 \mathcal{J}_1 \qquad & \mathcal{J}_1 &= K_1^2 + K_2^2 + \omega_2 J^2 \\ \mathcal{C}'_2 &= \mathcal{C}_2 & \mathcal{J}_2 &= 0. \end{aligned} \tag{4.4}$$

Hence the linear combination (3.2) has a single term: $\mathcal{J} = \alpha_1 \mathcal{J}_1$. The new generators coming from (3.3) read

$$\begin{aligned} &K'_1 = K_1 \qquad K'_2 = K_2 \qquad J' = J \\ &H' = 2\alpha_1(K_1P_1 + K_2P_2 + \omega_2 H) \\ &P'_1 = 2\omega_2\alpha_1(JP_2 - K_1H + P_1) \\ &P'_2 = 2\omega_2\alpha_1(-JP_1 - K_2H + P_2). \end{aligned} \tag{4.5}$$

In this case, the decomposition $g = t \oplus k$ coincides with the Cartan decomposition $g = p^{(1)} \oplus h^{(1)}$, and the three generators which are unchanged by the expansion close the isotropy subalgebra of a point/event $h^{(1)}$ (2.4). Taking into account (2.5) it is clear that proposition 1 can be applied. The expansion depends on a single parameter α_1, whose value (if the expansion indeed exists) is obtained by enforcing (4.3) for the three commutators $[P'_1, P'_2]$, $[H', P'_i]$. Let us compute, for instance,

$$\begin{aligned} [H', P'_1] &= 4\omega_2\alpha_1^2(-\omega_2 K_1 H^2 - K_1 P_1^2 - K_1 P_2^2) \\ &= -4\omega_2\alpha_1^2 K_1(\omega_2 H^2 + P_1^2 + P_2^2) = -4\omega_2\alpha_1^2 K'_1 \mathcal{C}_1. \end{aligned} \tag{4.6}$$

Note the automatic appearance of the Casimir $\mathcal{C}_1$; this will happen in all expansions we will deal with. Since the commutator must be equal to $\omega_1 K'_1$ we have

$$\alpha_1^2 = -\frac{\omega_1}{4\omega_2\mathcal{C}_1}. \tag{4.7}$$

It can be checked that the two remaining Lie brackets lead to the same condition.

Note that α_1 is not strictly speaking a number, but depends on the generators of the initial algebra *only* through the Casimir $\mathcal{C}_1$. Within any irreducible representation of the initial algebra, α_1 turns into a scalar value.

According to the different values for the initial constant $\omega_2 \neq 0$ (note that we start from $\omega_1 = 0$) and the possible choices of the expansion parameter α_1 (i.e. of the final ω_1), the process just described leads to the algebras displayed in the diagram:

$$\begin{array}{ccccccc} & so(4) & & iso(3) & & so(3,1) \\ \omega_2 > 0 & (+,+) & \longleftarrow & (0,+) & \longrightarrow & (-,+) \\ & \text{Elliptic} & & \text{Euclidean} & & \text{Hyperbolic} \\ & so(2,2) & & iso(2,1) & & so(3,1) \\ \omega_2 < 0 & (+,-) & \longleftarrow & (0,-) & \longrightarrow & (-,-) \\ & \text{Anti-de Sitter} & & \text{Poincaré} & & \text{de Sitter} \end{array}$$

This type of expansion allows us to 'recover' a space of constant curvature (elliptic/hyperbolic, or anti de Sitter/de Sitter) out of a flat space, either the 3D Euclidean space or the $(2+1)$D Minkowskian spacetime.

4.2. From extended Galilei to Newton–Hooke

In the non-generic case $\omega_2 = 0$, we must start the ω_1 expansion from the degenerate Galilei algebra. We want to keep $\omega_2 = 0$ but to introduce $\omega_1 \neq 0$, then reaching the Newton–Hooke algebras. The commutators which are zero in the initial algebra but not in the expanded one are only

$$[H, P_i] = 0. \tag{4.8}$$

The Galilean Casimirs read

$$\mathcal{C}_1 = P_1^2 + P_2^2 \qquad \mathcal{C}_2 = -P_1 K_2 + P_2 K_1. \tag{4.9}$$

We split the Newton–Hooke invariants as

$$\begin{aligned} \mathcal{C}_1' &= \mathcal{C}_1 + \omega_1 \mathcal{J}_1 \qquad & \mathcal{J}_1 &= K_1^2 + K_2^2 \\ \mathcal{C}_2' &= \mathcal{C}_2 & \mathcal{J}_2 &= 0. \end{aligned} \tag{4.10}$$

Thus $\mathcal{J} = \alpha_1 \mathcal{J}_1$. Should we apply the expansion recipe blindly, from (3.3) we obtain that $H' = 2\alpha_1(K_1 P_1 + K_2 P_2)$, all other generators being unchanged. Although proposition 1 cannot be used in this case to shortcut computations (note that $[k, t] \subset k$), it can be checked that the new generators thus obtained do close a Lie algebra, which is, however, *not* within the set of the algebras described in (2.1). In this case the initial Lie algebra is too much contracted (or Abelian) to be able to act as a germ for an expansion to the Newton–Hooke algebras. However, this problem can be circumvented in the same way as in the (1 + 1)-dimensional case [13]: starting not from Galilei algebra itself, but from a central extension, with central generator Ξ and characterized by a parameter m, the mass of a free particle. The Lie brackets of this extended Galilei algebra are given by

$$\begin{array}{lll} [J, P_i] = \epsilon_{ij} P_j \qquad & [J, K_i] = \epsilon_{ij} K_j \qquad & [J, H] = 0 \\ [P_1, P_2] = 0 & [K_1, K_2] = 0 & [P_i, K_j] = \delta_{ij} m \Xi \\ [H, P_i] = 0 & [H, K_i] = -P_i & [\Xi, \cdot\,] = 0. \end{array} \tag{4.11}$$

We keep $\mathcal{J} = \alpha_1(K_1^2 + K_2^2)$ and apply again the recipe (3.3) to define the expanded generators; due to the presence of the central extension the results found formerly change, and now we obtain

$$\begin{aligned} & K_1' = K_1 \qquad K_2' = K_2 \qquad J' = J \\ & H' = 2\alpha_1(K_1 P_1 + K_2 P_2 + m\Xi) \\ & P_1' = -2\alpha_1 m \Xi K_1 \qquad P_2' = -2\alpha_1 m \Xi K_2. \end{aligned} \tag{4.12}$$

Hence the subalgebra k unchanged by the expansion coincides with $h^{(1)}$, the isotropy subalgebra of an event. In spite of the central extension, the same reasoning as proposition 1 shows that the Lie brackets $[k', k']$ and $[k', t']$ are kept in the same form as in the non-extended initial Galilei algebra. The remaining commutators $[t', t']$ lead to

$$[H', P_i'] = -4\alpha_1^2 m^2 \Xi^2 K_i \equiv \omega_1 K_i' \qquad [P_1', P_2'] = 0 \tag{4.13}$$

and consequently

$$\alpha_1^2 = -\frac{\omega_1}{4m^2\Xi^2}. \tag{4.14}$$

This Galilean expansion recovers a non-zero curvature ω_1 out of the flat Galilei spacetime, while keeping $\omega_2 = 0$ which is accompanied by the presence of 'absolute time', producing the two curved 'absolute time' Newton–Hooke spacetimes and thereby completing the non-generic missing middle line in the diagram of section 4.1:

$$\begin{array}{lccccc} & t_4(so(2) \oplus so(2)) & & \overline{iiso}(1,1) & & t_4(so(2) \oplus so(1,1)) \\ \omega_2 = 0 & (+, 0) & \longleftarrow & (0, 0) & \longrightarrow & (-, 0) \\ & \text{Oscillating NH} & & \text{Extended Galilei} & & \text{Expanding NH} \end{array}$$

5. Speedspace expansions or ω_2 expansions

In this section, we switch roles for ω_1 and ω_2, and we discuss expansions which starting from the algebra with $\omega_2 = 0$ 'introduce' a non-zero value for the constant ω_2, the value of ω_1 being unchanged. Again some details are slightly different according to either $\omega_1 = 0$ or $\omega_1 \neq 0$, so we will study these separately. The name speedspace we give to these expansions is justified because when applied to the kinematical algebras, these expansions lead from the Galilei algebra to the Poincaré or to the 3D Euclidean one, while from Newton–Hooke the expansion leads either to the two de Sitter algebras, or to the 3D elliptic and hyperbolic algebras.

5.1. From Newton–Hooke to de Sitter

We consider as the initial algebras those with $\omega_2 = 0$ and a fixed $\omega_1 \neq 0$, that is, the Newton–Hooke ones. There are four Lie brackets of (2.1) which are zero in the initial algebra but should be different from zero in the expanded one:

$$[P_1, P_2] = 0 \qquad [K_1, K_2] = 0 \qquad [P_1, K_1] = 0 \qquad [P_2, K_2] = 0. \tag{5.1}$$

The two Casimirs (2.2) are now

$$\mathcal{C}_1 = P_1^2 + P_2^2 + \omega_1(K_1^2 + K_2^2) \qquad \mathcal{C}_2 = -P_1K_2 + P_2K_1. \tag{5.2}$$

We decompose the two Casimir invariants of the algebras we want to reach by expansion (isomorphic to either $so(3)$ or $so(2,1)$) by taking into account the expansion constant ω_2:

$$\begin{aligned} \mathcal{C}_1' &= \mathcal{C}_1 + \omega_2\mathcal{J}_1 \qquad \mathcal{J}_1 = H^2 + \omega_1 J^2 \\ \mathcal{C}_2' &= \mathcal{C}_2 + \omega_2\mathcal{J}_2 \qquad \mathcal{J}_2 = JH. \end{aligned} \tag{5.3}$$

Therefore, the element (3.2) has two terms and gives rise to the new generators defined by

$$\begin{aligned} &H' = H \qquad J' = J \\ &P_1' = 2\omega_1\alpha_1(K_1H + JP_2) + \alpha_2(P_2H + \omega_1JK_1) \\ &P_2' = 2\omega_1\alpha_1(K_2H - JP_1) + \alpha_2(-P_1H + \omega_1JK_2) \\ &K_1' = 2\alpha_1(-P_1H + \omega_1JK_2) + \alpha_2(K_2H - JP_1) \\ &K_2' = 2\alpha_1(-P_2H - \omega_1JK_1) + \alpha_2(-K_1H - JP_2). \end{aligned} \tag{5.4}$$

In this case the decomposition $g = t \oplus k$ coincides with the Cartan one associated with the involution $\mathcal{P}$, and the invariant generators H and J generate the isotropy subalgebra $h^{(2)}$ of a (timelike) line (2.4). This means that proposition 1 can be applied. Thus we have only to compute the Lie brackets involving the four generators P_i', K_i'. Let us choose, for instance,

$$\begin{aligned} [P_1', P_2'] = &-4\omega_1^2\alpha_1^2(2K_1P_2H - 2K_2P_1H + JP_1^2 + JP_2^2 + \omega_1JK_1^2 + \omega_1JK_2^2) \\ &-\omega_1\alpha_2^2(2K_1P_2H - 2K_2P_1H + JP_1^2 + JP_2^2 + \omega_1JK_1^2 + \omega_1JK_2^2) \\ &-2\omega_1\alpha_1\alpha_2(2P_1^2H + 2P_2^2H + 2\omega_1K_1^2H + 2\omega_1K_2^2H \\ &+4\omega_1JK_1P_2 - 4\omega_1JK_2P_1). \end{aligned} \tag{5.5}$$

We introduce in this expression the Newton–Hooke Casimirs (5.2) and we obtain

$$\begin{aligned} [P_1', P_2'] &= -4\omega_1^2\alpha_1^2(2\mathcal{C}_2H + J\mathcal{C}_1) - \omega_1\alpha_2^2(2\mathcal{C}_2H + J\mathcal{C}_1) - 2\omega_1\alpha_1\alpha_2(2\mathcal{C}_1H + 4\omega_1J\mathcal{C}_2) \\ &= -(8\omega_1^2\mathcal{C}_2\alpha_1^2 + 2\omega_1\mathcal{C}_2\alpha_2^2 + 4\omega_1\mathcal{C}_1\alpha_1\alpha_2)H' - (4\omega_1^2\mathcal{C}_1\alpha_1^2 + \omega_1\mathcal{C}_1\alpha_2^2 + 8\omega_1^2\mathcal{C}_2\alpha_1\alpha_2)J' \end{aligned} \tag{5.6}$$

and by imposing (5.6) to be equal to $\omega_1\omega_2 J'$ we obtain two quadratic equations in the constants α_l:

$$\begin{aligned} &4\omega_1\mathcal{C}_1\alpha_1^2 + \mathcal{C}_1\alpha_2^2 + 8\omega_1\mathcal{C}_2\alpha_1\alpha_2 = -\omega_2 \\ &4\omega_1\mathcal{C}_2\alpha_1^2 + \mathcal{C}_2\alpha_2^2 + 2\mathcal{C}_1\alpha_1\alpha_2 = 0. \end{aligned} \tag{5.7}$$

If we calculate any other Lie bracket (5.1) with the new generators (5.4) we obtain the same equations (5.7). Moreover, we also have to compute the commutators $[P_1', K_2']$ and $[P_2', K_1']$; they are directly zero and do not originate any relation for the constants α_l.

Hence, within an irreducible representation of the initial algebra, the Casimirs appear replaced by their eigenvalues, and the solutions in α_1 and α_2 for the quadratic equations (5.7) afford the expansions we are looking for.

These expansions which start from the Newton–Hooke algebras introduce the constant ω_2 in the four-dimensional spaces of lines $\mathcal{S}^{(2)}$ ($\omega_2 = -1/c^2$ when it is negative), thus eliminating the 'absolute time' character and giving rise to the curved relativistic de Sitter algebras; they embrace the following cases:

$$\begin{array}{cc} \omega_1 > 0 & \omega_1 < 0 \\ & \\ so(4) & so(3,1) \\ (+,+) & (-,+) \\ \text{Elliptic} & \text{Hyperbolic} \\ \uparrow & \uparrow \\ t_4(so(2)\oplus so(2)) & t_4(so(2)\oplus so(1,1)) \\ (+,0) & (-,0) \\ \text{Oscillating NH} & \text{Expanding NH} \\ \downarrow & \downarrow \\ so(2,2) & so(3,1) \\ (+,-) & (-,-) \\ \text{Anti-de Sitter} & \text{de Sitter} \end{array}$$

5.2. From Galilei to Poincaré

Finally, we consider the ω_2 expansion starting from the Galilei algebra which has not only $\omega_2 = 0$ but also $\omega_1 = 0$. We want to obtain Lie algebras with $\omega_2 \neq 0$, but keeping the Galilean value of ω_1. The Lie brackets that we have to make different from zero read (see equation (2.1)):

$$[K_1, K_2] = 0 \qquad [P_1, K_1] = 0 \qquad [P_2, K_2] = 0. \tag{5.8}$$

By taking into account the Galilean Casimirs (4.9) we write the invariants (2.2) with $\omega_1 = 0$ as

$$\begin{aligned} &\mathcal{C}_1' = \mathcal{C}_1 + \omega_2\mathcal{J}_1 \qquad \mathcal{J}_1 = H^2 \\ &\mathcal{C}_2' = \mathcal{C}_2 + \omega_2\mathcal{J}_2 \qquad \mathcal{J}_2 = JH. \end{aligned} \tag{5.9}$$

Therefore, the generators for the expanded algebras are

$$\begin{aligned} &H' = H \qquad J' = J \\ &P_1' = \alpha_2 P_2 H \qquad P_2' = -\alpha_2 P_1 H \\ &K_1' = -2\alpha_1 P_1 H + \alpha_2(K_2 H - J P_1) \\ &K_2' = -2\alpha_1 P_2 H - \alpha_2(K_1 H + J P_2). \end{aligned} \tag{5.10}$$

As in the previous expansion, we have only to compute the commutators between the generators P_i', K_i'. Enforcing the values they should have in the expanded algebra we obtain the constants α_l:

$$
\begin{aligned}
&[P_i', K_i'] = -\alpha_2^2(P_1^2 H + P_2^2 H) = -\alpha_2^2 \mathcal{C}_1 H \equiv \omega_2 H' \\
&[K_1', K_2'] = -\alpha_2^2(2K_1P_2H - 2K_2P_1H + JP_1^2 + JP_2^2) - 4\alpha_1\alpha_2(P_1^2 + P_2^2)H \\
&\qquad\qquad = -2\alpha_2(2\alpha_1\mathcal{C}_1 + \alpha_2\mathcal{C}_2)H - \alpha_2^2\mathcal{C}_1 J \equiv \omega_2 J' \\
&[P_1', P_2'] = 0 \qquad [P_1', K_2'] = 0 \qquad [P_2', K_1'] = 0
\end{aligned}
\tag{5.11}
$$

that is,

$$
\alpha_2^2 = -\frac{\omega_2}{\mathcal{C}_1} \qquad \alpha_1 = -\frac{\alpha_2\mathcal{C}_2}{2\mathcal{C}_1}. \tag{5.12}
$$

This Galilean expansion which recovers the curvature ω_2 of the space of (timelike) lines $\mathcal{S}^{(2)}$ gives rise to the Euclidean and Poincaré algebras, and involve the eigenvalue of both Casimirs:

$$
\begin{array}{c}
\omega_1 = 0 \\[1em]
iso(3) \\
(0,+) \\
\text{Euclidean} \\
\uparrow \\
iiso(2) \\
(0,0) \\
\text{Galilei} \\
\downarrow \\
iso(2,1) \\
(0,-) \\
\text{Poincaré}
\end{array}
$$

6. Concluding remarks

We have presented an expansion method which allows us to reverse all the contraction arrows of the Lie algebras displayed in table 1. We would like to stress several points which turn out to be relevant, and which may hint towards the still rather unknown extension to the expansion procedure to either higher-dimensional situations or to higher-rank cases.

First, in the ω_1 expansions recovering the curvature ω_1 of the space $\mathcal{S}^{(1)}$ only the first Casimir $\mathcal{C}_1$ appears, while both Casimirs participate in the ω_2 expansions making different from zero the curvature ω_2 of the space of lines $\mathcal{S}^{(2)}$. As the ranks of the homogeneous spaces $\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)}$ are one and two, respectively, the results obtained seem to confirm the expected relationship between the rank of the space and the number of Casimirs needed to perform the expansion. This idea is in agreement with the known generalizations to arbitrary dimension and variants of expansions $iso(p,q) \to so(p,q)$ (associated with $\mathcal{S}^{(1)}$) [7, 8] which only involve a single Casimir (the quadratic one) and are in a sense direct generalizations to any higher dimension from the ω_1 expansions we discuss here.

Secondly, the role of extended algebras as the starting point for the expansion also needs clarification. This role clearly depends on the type of expansion to be done. While the starting point for the ω_1 expansion of Galilei algebra should be an extended Galilei algebra, this is not necessary for the ω_2 expansion of the same algebra. A complete and systematic study of all the central extensions of the quasi-orthogonal algebras is available [14], and should be the starting point for understanding the role these extensions play in the expansion process, a problem which deserves further study.

However, the more interesting open question would be to know whether or not some suitable 'extension' of the method we have proposed is still applicable for higher dimensions. It is natural to suppose that whatever the correct method should be, it should rest again on the Casimirs of the initial and the expanded algebra, and their dependence on the expansion constant ω_a. Two facts will likely complicate the issue under discussion. First, further to the quadratic Casimir, the additional ones are higher order (this is masked in the $so(4)$ family because the additional Casimir here is a perfect square and it can be considered as an extra quadratic one). Secondly, the dependence of higher-order Casimirs on the expansion constant ω_a is also known in the general case [15] and this dependence is not only linear but also given by a higher-order polynomial. The analysis of the next situation, the $(3+1)$-dimensional case, would help to clarify the above questions.

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